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LETTER TO THE EDITOR

Series analysis for the three-state Potts model

I G Enting

Physics Department, Northeastern University, Boston, Massachusetts 02115, USA

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Abstract. Low-temperature series expansions for the zero-field partition function and order parameter of the three-state Potts model have been extended to u^{31}. Estimates for α' and β show rather poor convergence but are consistent with the values $\alpha' = \frac{1}{3}$, $\beta = \frac{1}{9}$ which characterise the 'hard hexagons' lattice gas.

The problem of determining the critical exponents of the three-state Potts model in two dimensions has proved particularly difficult. Both series analysis and renormalisation group techniques have been used, producing a number of conflicting estimates.

The question of the exponents is of current interest for two reasons:

(i) den Nijs (1979) has conjectured that $\alpha = \alpha' = \frac{1}{3}$. This conjecture is a special case of a relation that serves to unify critical and tricritical points (Nienhuis *et al* 1979) and so it is particularly interesting.

(ii) Baxter (1980) has obtained an exact solution for the 'hard hexagons' problem and finds that $\alpha = \frac{1}{3}$, $\beta = \frac{1}{9}$. The hard hexagons system is a lattice gas on the triangular lattice with nearest-neighbour exclusion. Alexander (1975) suggested that this type of system should have the same exponents as the three-state Potts model.

In order to test these ideas, low-temperature series for the reduced partition function Z and the order parameter M have been extended to order u^{31} (for zero field) using the finite-lattice method (de Neef and Enting 1977) in the manner described by Enting (1978).

The series are of the form:

$$Z = 1 + \sum_{n=4}^{\infty} a_n u^n = 1 + 2u^4 + 4u^6 + \ldots$$

$$M = 1 - \sum_{n=4}^{\infty} b_n u^n = 1 - 3u^4 - 12u^6 - \ldots.$$

The coefficients a_n, b_n, $n \leqslant 31$ are given in table 1.

The analysis of these series shows that the convergence of the exponent estimates is rather poor. This may well be due to significant correction terms which modify the assumed power-law behaviour.

Among the many exponent estimates, those of Zwansig and Ramshaw (1977) giving $\alpha' = 0{\cdot}296 \pm 0{\cdot}002$ and Miyashita *et al* (1979) giving $\beta = 0{\cdot}1064 \pm 0{\cdot}0005$ are of particular significance because of the high precision that was claimed.

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Table 1. Coefficients in the expansion of Z and M

$$Z = 1 + \sum_{n=4}^{\infty} a_n u^n, \qquad M = 1 - \sum_{n=4}^{\infty} b_n u^n.$$

n	a_n	b_n
4	2	3
5	0	0
6	4	12
7	4	12
8	6	36
9	24	108
10	24	210
11	68	480
12	190	1746
13	192	2340
14	904	10 566
15	1420	19 500
16	3160	53 976
17	9940	152 604
18	14 572	329 424
19	49 268	971 304
20	102 886	2403 291
21	225 004	5955 576
22	652 940	16 858 584
23	1301 256	40 337 376
24	3513 806	110 301 321
25	8591 792	287 061 696
26	19 326 248	730 223 208
27	52 781 148	1985 703 720
28	120 709 472	5070 001 716
29	306 339 824	13 446 444 720
30	779 682 608	35 650 214 232
31	1852 672 272	92 442 918 828

Zwansig and Ramshaw based their analysis on a Neville table extrapolation in which the second-order extrapolants were steadily increasing while the third-order extrapolants were steadily decreasing. In actual fact, the trend in the second-order extrapolants is reversed by the u^{17} term and for $n > 20$, all the first- to fifth-order extrapolants (estimates of $2 - \alpha'/2$) show a steady decline.

By order 31 the estimate of α has increased to about 0·31 but the slow convergence suggests that the Neville tables are not the best way to extrapolate the series. The value $\alpha = \alpha' = \frac{1}{3}$ is quite consistent with the trends observed for $n > 20$.

In an attempt to obtain α' by an alternative means, series for the energy were analysed, exploiting the fact that the critical energy is known exactly (Kihara *et al* 1954).

The function analysed was E_s where

$$E_s = U_c - U = (1 - \sqrt{\tfrac{1}{3}}) - u(\mathrm{d}/\mathrm{d}u) \ln Z.$$

The expected behaviour is $E_s \sim (u_c - u)^{1-\alpha'}$ so Padé approximants to $(u_c - u)(\mathrm{d}/\mathrm{d}u) \ln E_s$ should give estimates of $\alpha' - 1$. These estimates are given in table 2 and they show a steady trend towards smaller values of α' as the number of series terms

Table 2. Exponent estimates for $1-\alpha$ and β obtained by evaluating Padé approximants to $(u_c-u)(\mathrm{d}/\mathrm{d}u)\ln(U_c-U)$ and $(u_c-u)(\mathrm{d}/\mathrm{d}u)\ln M$ at $u=u_c$.

$[N, M]$	Estimates of $1-\alpha$	Estimates of β
8, 7	0·6032	—
7, 8	0·6036	—
8, 8	0·6068	0·107 09
9, 8	0·6344	0·106 75
8, 9	1·0883	0·107 05
9, 9	0·6176	0·105 64
10, 9	0·6160	0·107 86
9, 10	0·6161	0·109 55
10, 10	0·6162	0·108 42
11, 10	0·6153	0·108 61
10, 11	0·6160	0·108 55
11, 11	0·6159	0·108 65
12, 11	0·6156	0·108 60
11, 12	0·6160	0·108 36
12, 12	0·6180	0·112 75
13, 12	0·6289	0·109 13
12, 13	0·5462	0·108 96
13, 13	0·6229	0·109 16
14, 13	0·6252	0·109 12
13, 14	0·6267	0·109 30
14, 14	0·6279	0·109 27
15, 14	0·6251	0·110 06
14, 15	0·6267	0·109 30
15, 15	—	0·106 85

increases. Again this trend is consistent with $\alpha'=\frac{1}{3}$ but the rate of convergence is disappointingly slow.

The order parameter series was analysed by constructing Padé approximants to $(u_c-u)(\mathrm{d}/\mathrm{d}u)\ln M$ and evaluating them at $u=u_c=(\sqrt{3}-1)/2$ to give estimates of β. These estimates are listed in table 2. The estimates using short series are in agreement with the value 0·1064 obtained by Miyashita *et al* but as more series terms are included the estimates increase steadily to about 0·109. Again the trend is consistent with the hard hexagons exponent.

Watts (1974) pointed out the possibility of extrapolating exponent estimates from $[M, N]$ Padé approximants by plotting them against $(M+N)^{-1}$. When the α', β estimates from table 2 are plotted in this way the results are consistent with linear trends towards $\frac{1}{3}$ and $\frac{1}{9}$ respectively, but the scatter in the α' estimates (and to a lesser extent in the β estimates) is too large to allow for useful extrapolations.

The surprising aspect of the 'hard hexagons' exponents is that scaling gives $\delta=14$. If the value $\delta=14$ did apply to Potts models it would indicate that the errors in series estimates for δ had been underestimated in the same way that errors in α, β were underestimated. If δ did vary with q, the number of states in the Potts model, then the $q=1$ case (bond percolation) which as $\delta\approx 18$ (Gaunt and Sykes 1976) would no longer appear as an anomaly.

A more extensive analysis of these series is currently in progress. Unfortunately the finite lattice method is not a particularly efficient technique for obtaining high-field expansions and so a direct test of the $\delta=14$ conjecture may not be possible.

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References

Alexander S 1975 *Phys. Lett.* **54**A 353–4
Baxter R J 1980 *J. Phys. A: Math. Gen.* **13** L61–70
Enting I G 1978 *Aust. J. Phys.* **31** 515–22
Gaunt D S and Sykes M F 1976 *J. Phys. A: Math. Gen.* **9** 1109–16
Kihara T, Midzuno Y and Shizume T 1954 *J. Phys. Soc. Japan* **9** 681–7
Miyashita S, Betts D D and Elliott C J 1979 *J. Phys. A: Math. Gen.* **12** 1605–22
de Neef T and Enting I G 1977 *J. Phys. A: Math. Gen.* **10** 801–5
Nienhuis B, Berker A N, Riedel E K and Schick M 1979 *Phys. Rev. Lett.* **43** 737–40
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857–68
Watts M G 1974 *J. Phys. A: Math. Gen.* **7** L85–8
Zwansig B and Ramshaw J D 1977 *J. Phys. A: Math. Gen.* **10** 65–9